

Searching a Database under Decoherence

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We report on the effects of a simple decoherence model on the quantum search algorithm. Despite its simplicity, the decoherence model is an instructive model that can genuinely imitate realistic noisy environment in many situations. As one would expect, as the size of the database gets larger, the effects of decoherence on the efficiency of the quantum search algorithm cannot be ignored. Moreover, with decoherence, it may not be useful to iterate beyond the first maxima in the probability distribution of the search entry. Surprisingly, we also find that the number of iterations for maximum probability of the search entry reduces with decoherence.

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I. INTRODUCTION

In many computations, the search algorithm is one of the most time-consuming activity of the computer program [1]. With the rapid proliferation of the internet and the increasing need to retrieve information from this ever expanding internet network, it is crucial to consider alternatives to current search algorithms. Recently, quantum algorithms have been shown to reduce hitherto computationally difficult problems for which there are no known polynomial time algorithm into a tractable problem involving polynomial time [2]. This problem involves factoring a large composite integer into its prime factors. Besides prime number factorization, it has also been shown that quantum algorithms like the quantum search algorithm could substantially improve the searching process for a database [3–5] despite only a square root speed-up.

Indeed, in recent years, this immense potential in quantum algorithm has spurred a fecundity of ideas on the actual implementation of a quantum computer. However, the prospect of actually building one in the next decade remains taunting and seemingly insurmountable with current technology due primarily several reasons, like decoherence and scalability. Unlike classical bits, quantum bits (qubits) are highly susceptible to collapse due to the difficulty of isolating the quantum mechanical systems from its environment. Such decoherence inevitably leads to a loss of information within the system. Thus it is necessary to consider fault tolerant computation through quantum error correction [6] or more recently using decoherence-free subspaces [7, 8].

Recently, some interesting works have been done to consider the robustness of the quantum search algo-

rithm under a noisy environment [9, 10]. In ref. [9], they modeled the decoherence of the search algorithm using a stochastic white noise. In this paper, by considering a specific decoherence model, we obtain analytic expression for the probability of the search item in a quantum search algorithm after a certain number of iterations. Our analytic expression can certainly facilitate the study of the behavior and efficiency of the search algorithm within the model.

This paper is organized as follows. In section II, we briefly describe the Grover's search algorithm under a noise-free environment. In section III, we describe our decoherence model and in section IV derive an explicit analytical formula for the probability of the search item after a certain number of iterations within the model. In section V, we consider the robustness of the search algorithm under the decoherence model. Finally in section VI, we summarize our results and discuss some implications of the model.

II. GROVER'S SEARCH

In a series of seminal papers, Grover [3–5] considered a quantum algorithm that could achieve a speed-up in the computational implementation with only $O(\sqrt{N})$ for a large structured database with N records. This algorithm compares favorably with the classical result which can only execute a search with $O(N)$ efficiency. Moreover, it has been shown that Grover's algorithm is optimal [11].

Grover's search algorithm can be summarized neatly into the following main steps: (i) Initialization of the system into a superposition of states; (ii) Subjecting the system to a hashing function, $C(S)$,

represented by a unitary operator, U , given by

$$U = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (1)$$

(assuming that the first entry satisfies the search criteria and therefore undergoes a rotation of π radians) followed by a diffusion matrix, D , defined by

$$D_{ij} = -\delta_{ij} + \frac{2}{N} \quad (2)$$

for $O(\sqrt{N})$ iterations; (iii) Measuring the resulting state. The heart of the process therefore hinges significantly on the on step (ii) and the $O(\sqrt{N})$ iterations of the matrix $DU \equiv S$. In this paper, we look closely into this important step and scrutinized the ideas behind the efficiency.

Before proceeding further, we first consider the eigenvalues of the matrix S . It is not difficult to show that the eigenvalues of S all lie on the locus $|z| = 1$, unit circle, on the complex Argand plane and are explicitly $\underbrace{\{-1, \dots, -1, \eta, \eta^*\}}_{(N-2)}$ where the root η and η^*

satisfy the equation $z^2 - \frac{2(N-2)}{N}z + 1 = 0$. Indeed, it can be further shown that if the matrix S is diagonalized as $P\Lambda P^{-1}$, with $\Lambda = \text{Diag}[\underbrace{-1, \dots, -1}_{(N-2)}, \eta, \eta^*]$

and

$$\eta = \frac{1}{N} (N - 2 - 2i\sqrt{N-1}) \quad (3)$$

then the matrix P assumes the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & i\sqrt{N-1} & -i\sqrt{N-1} \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}. \quad (4)$$

Since $|\eta| = 1$, we can rewrite the complex number η as $e^{-i\theta}$, where, using eq(3),

$$\cos \theta = \frac{N-2}{N} \quad \text{and} \quad \sin \theta = \frac{2\sqrt{N-1}}{N}. \quad (5)$$

Finally, we note that the probability of finding the search item can be evaluated to be $\frac{1}{N}(\cos(m\theta) + \sqrt{N-1}\sin(m\theta))^2$ while the probability of getting one of the other entries in the database is $\frac{1}{N}(\cos(m\theta) - \frac{1}{\sqrt{N-1}}\sin(m\theta))^2$.

III. NOISE INDUCED THROUGH DECOHERENCE

A. Superoperators

Decoherence can be studied and understood within the context of superoperators through the evolution of a bipartite quantum system [12]. A superoperator describes a linear map, $\$$, from input density matrix, ρ_{in} , to output density matrix, ρ_{out} . Although the superoperator, $\$$, is not a unitary operator, it is a linear map that preserves hermiticity as well as the trace.

The standard procedure of understanding the behavior of one part of a bipartite quantum system is to extend the system to a larger one (in which the environment (E) is incorporated) so that the evolution of state becomes unitary under the transformation, \mathcal{U} . By assuming complete positivity of the superoperators, it is possible to study the non-unitarity evolution of the state of a subsystem using an operator sum representation. In terms of the operator sum or Kraus representation, we can express this map, $\$$, more succinctly as

$$\rho_{\text{out}} = \$(\rho_{\text{in}}) = \sum_{\mu} M_{\mu} \rho_{\text{in}} M_{\mu}^{\dagger}. \quad (6)$$

Unitarity of the operator, \mathcal{U}_{CE} , also requires that the Kraus operators satisfy the condition

$$\sum_{\mu} M_{\mu} M_{\mu}^{\dagger} = 1 \quad (7)$$

B. Decoherence Model

For two-spin-1/2 system, we can consider the following Kraus representation

$$M_0 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \quad (8)$$

$$M_1 = \begin{pmatrix} 0 & \sqrt{p/3} \\ \sqrt{p/3} & 0 \end{pmatrix} = \sqrt{p/3} \sigma^x \quad (9)$$

$$M_2 = \begin{pmatrix} 0 & -\sqrt{p/3} \\ \sqrt{p/3} & 0 \end{pmatrix} = \sqrt{p/3} i \sigma^y \quad (10)$$

$$M_3 = \begin{pmatrix} \sqrt{p/3} & 0 \\ 0 & -\sqrt{p/3} \end{pmatrix} = \sqrt{p/3} \sigma^z \quad (11)$$

where σ^i , $i = x, y, z$ are the usual Pauli matrices. We can check easily that the unitarity condition for the larger space is satisfied since $\sum_{\mu} M_{\mu}^{\dagger} M_{\mu} = 1$. This

decoherence model is sometimes alluded to as the depolarizing channel.

This decoherence model is an idealization of a transmission and storage process [13] with some elegant mathematical symmetry in which the quantum state in the channel can undergo a bit-flip and phase errors. The construction of a depolarizing channel arises from the interaction of a quantum state, $|\Psi\rangle$ with an environment in which there is a probability p that the quantum state will survive and a probability $p/3$ that it would execute a pure bit-flip, a pure phase error or a combination of both. Thus, schematically, we have

$$|\Psi\rangle \xrightarrow{1-p} |\Psi\rangle \quad (12)$$

$$|\Psi\rangle \xrightarrow{p/3} \sigma_1 |\Psi\rangle \quad (13)$$

$$|\Psi\rangle \xrightarrow{p/3} \sigma_3 |\Psi\rangle \quad (14)$$

$$|\Psi\rangle \xrightarrow{p/3} \sigma_2 |\Psi\rangle \quad (15)$$

Furthermore, it is possible to show that for general spin-1/2 particle in which the density matrix describing the mixed states is given by $\rho_0 = 1/2(1 + \vec{n} \cdot \sigma)$, that the original density matrix ρ_0 evolves under the depolarizing channel as

$$\rho' = p \frac{1}{2} \mathbf{I}_2 + (1-p)\rho_0 \quad (16)$$

where \mathbf{I}_2 is the 2×2 unit matrix. In order to apply this model to the quantum search algorithm, it is necessary to consider the extension to N qubits. It is not hard to verify that in this case, Eq.(16) becomes

$$\rho' = p \frac{1}{N} \mathbf{I}_N + (1-p)\rho_0 \quad (17)$$

where \mathbf{I}_N is the $N \times N$ unit matrix.

IV. SEARCH WITH DECOHERENCE

Let us now study the effects of the decoherence (depolarizing) model to the quantum search algorithm. As in any search algorithm, we first initialize the system into a superposition of states with equal probabilities. Thus the initial density matrix of the system is

$$(\rho_0)_{ij} = \frac{1}{N}. \quad (18)$$

This state is then subject to the usual inversion-diffusion process so that the density matrix at the end of the first transformation is

$$\rho'_0 = S \rho_0 S^\dagger. \quad (19)$$

In the noise-free Grover's search algorithm, this transformation is repeated a certain number of times. In our model, before we perform the second iteration,

we allow the system to evolve under a depolarizing channel so that using Eq.(17), we have

$$\rho_{i+1} = p \frac{1}{N} \mathbf{I}_N + (1-p)\rho_i. \quad (20)$$

It is easy to show inductively that for m iterations,

$$\begin{aligned} \rho_m = & p \frac{1}{N} \mathbf{I}_N (1 + (1-p) + \dots \\ & + (1-p)^{m-1}) + (1-p)^m S^m \rho_0 S^{\dagger m} \rho_i. \end{aligned} \quad (21)$$

$$= \frac{1}{N} [1 - (1-p)^m] + (1-p)^m S^m \rho_0 S^{\dagger m} \quad (22)$$

A schematic representation of the model is shown in Fig. 1.

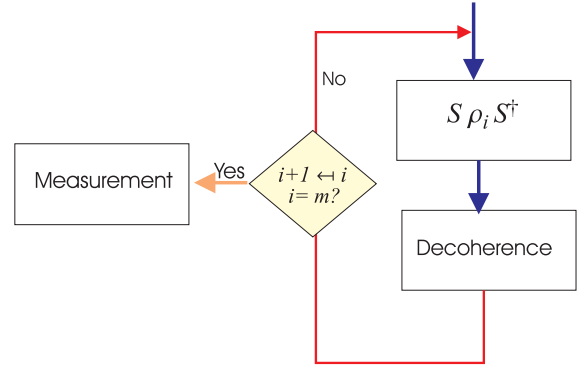


FIG. 1. Schematic illustration of a decoherence model.

After the m iterations, the state of the system is measured using the standard Von-Neumann measurement or positive operator valued measurement through appropriate extension to a higher dimensional space, so that the state after the measurement reads

$$\rho_f = \sum_i E_i \rho_m E_i^\dagger \quad (23)$$

where E_i are orthogonal projection operators (POVM) $|i\rangle\langle i|$. However, we note that $\sum_i E_i S^m \rho_0 S^{m\dagger} E_i^\dagger$ is just the probability of finding the desired state under the search without decoherence. Moreover, using the results in section II or ref. [9, 14], we have

$$\sum_i E_i S^m \rho_0 S^{m\dagger} E_i^\dagger = \frac{1}{N} (\cos m\theta + \sqrt{N-1} \sin m\theta)^2, \quad (24)$$

where $\cos \theta = 1 - \frac{2}{N}$ and $\sin \theta = \frac{2\sqrt{N-1}}{N}$. Finally, using Eq.(22) and manipulating using some simple algebra, we find that the probability of searching successfully for the desired state under decoherence is

$$P(m) = \frac{1}{N} (1 - (1-p)^m [(\cos m\theta + \sqrt{N-1} \sin m\theta)^2 + 1]) \quad (25)$$

Eq. (25) is the main result in our paper.

For the case of a generalized measurement, we can consider the set of r POVMs given by $F_i = \sum_{ij} \lambda_{ij} E_j$, $i = 1 \dots r, j = 1 \dots N$ where the parameters λ_{ij} must satisfy the unitarity condition $\sum_{i=1}^r \lambda_{ij} = 1$ for each j . For simplicity, we can choose $r = N$ and $\lambda_{ii} = (1 - \epsilon)$, $\lambda_{ij} = \frac{\epsilon}{r-1}$, $i \neq j$. Indeed, it is not difficult to imagine an experimental setup for these POVMs since they can be used to describe possible errors in the photon detectors. In this case, we find the probability of successfully searching for the desired state is

$$P(m) = (1 - \epsilon) P(m)_{\text{ortho}} + \frac{\epsilon}{N} (1 - (1-p)^m [(\cos m\theta + \frac{1}{\sqrt{N-1}} \sin m\theta)^2 + 1]) \quad (26)$$

where $P(m)_{\text{ortho}}$ is the probability of the search item under Von Neumann orthogonal measurements and given in Eq.(25).

V. ANALYSIS

Within the search algorithm, the number of iterations needed for the search is usually fixed by maximizing the probability of getting the desired state. Without any decoherence, the number of iterations m_{max} of the first maximum for large N is found to be

$$m_{\text{max}} = \frac{\pi\sqrt{N}}{4}. \quad (27)$$

With decoherence, from Eq.(25), the condition is imposed by the following transcendental equation,

$$\frac{m}{1-p} (1 - \frac{N}{2} (1 - \cos \zeta)) = \frac{N\zeta}{2m+1} \sin \zeta \quad (28)$$

where $\zeta = 2(2m+1)\theta$.

It is interesting to compare the probability of the search item as a function of the number of iterations, m , for the specific case for various values of p . Such a graph for $N = 128$ and $p = 0.01, 0.04$ and 0.083394 is shown in Fig. 2. The value of $p = p_c = 0.083394$ refers to the maximum allowed value of the decoherence parameter p subject to $m_{\text{max}} = \pi\sqrt{N}/4$ such that the probability of the search item do not fall below 0.5. Fig. 2 clearly shows that the presence of

decoherence tends to provide a decaying effect on the periodic probability distribution of the search item. In the noise-free situation, we have iterations in which the probability of search items reaches a periodic maximum without significant decay in the maximum probability. In the case of a noisy search, we see that there is gradual reduction in the maxima so that subsequent maxima are usually rendered useless in the search as the magnitude of decoherence gets larger.

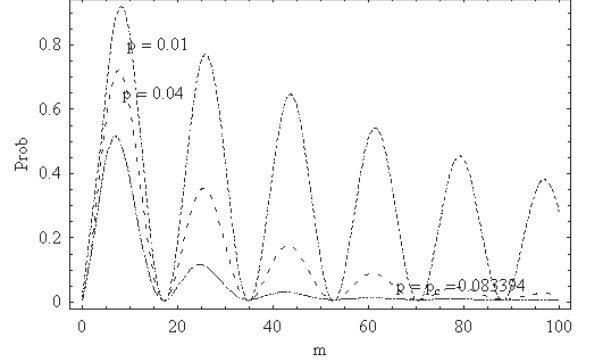


FIG. 2. Probability of search item as a function of the number of iterations for $p = 0.1, 0.4$ and 0.83394 and $N = 128$.

As we have noted earlier, if we maintain the number of iterations to the value required under the noise-free situation, namely m_{max} , we can solve Eq.(25) for the value of p_c such that the probability of the search item is above chance level (i.e. $P(m) > 0.5$) for $p < p_c$. In Fig 3, we plot the critical decoherence parameter p as a function of the database size $n = \log_2(N)$ up to a database size of about 2×10^6 . This graph shows that as the database gets larger, it is imperative and necessary to overcome the problem of decoherence. At a database size of $2^{21} \approx 2 \times 10^6$, the amount of decoherence allowed is only about 0.000609, which is a very small value indeed.

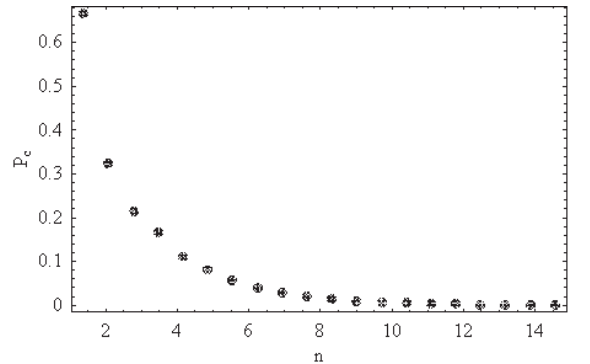


FIG. 3. Critical decoherence parameter p_c as a function of $n = \log_2(N)$

In a general search problem, the number of iterations can always be controlled. For a noisy search algorithm, it seems from our analysis that it may be better to reduce the number of iterations so that the

probability of the search items becomes a little larger. Using a database of $N = 1024$, we plot the variation of the probability of the search items as a function of iterations m for $p = 0.001, 0.014$ and $p = p_c = 0.0274$ in Fig. 4. The graph clearly shows a shift in the maxima for the curves. Indeed for the near critical value of $p = p_c$, we see that the maxima hovers around the value of 0.5 at $m = 21$. A three-dimensional plot of the probability as a function of m and p is shown in Fig 5. Indeed, by varying the decoherence parameter p , it is possible to solve for the value of m_{\max} corresponding to the probability of the search item. Fig. 6 shows the value of m_{\max} as a function of p . The step function refers to the integer part of m_{\max} .

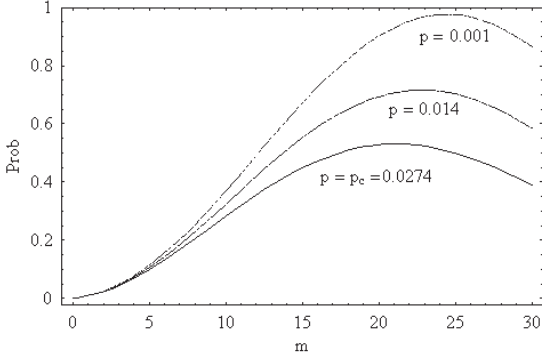


FIG. 4. Probability of the search item as a function of the number of iterations m for $p = 0.001, 0.014$ and $p = p_c = 0.0274$

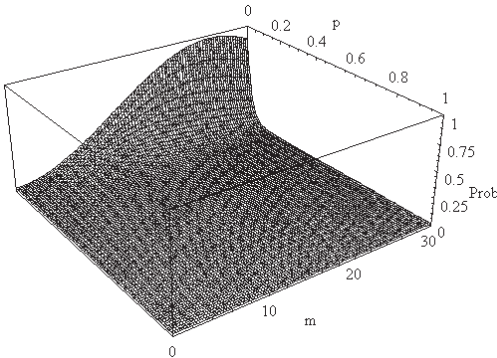


FIG. 5. Three-dimensional plot of the probability of the search item as a function of the number of iterations, m and decoherence parameter, p ($N = 1024$)

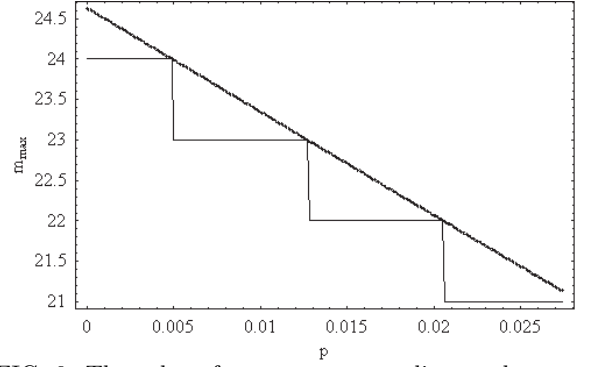


FIG. 6. The value of m_{\max} corresponding to the maximum probability of the search item as a function of p

It is interesting to note that the graph of m_{\max} corresponding to the maximum probability of the search item is a linear function of p . A interpolation fit gives $m_{\max} = 24.6254 - 127.7426p$. If we then plot the probability of the search item as a function of p using the integer part of this linear function, *i.e.* $\text{Int}(m_{\max})$ and compare it with the corresponding probability for $m_{\max} = \text{Int}(\pi\sqrt{N}/4)$, we see that the difference is reasonably negligible, especially for small p . This plot is illustrated in Fig. 7.

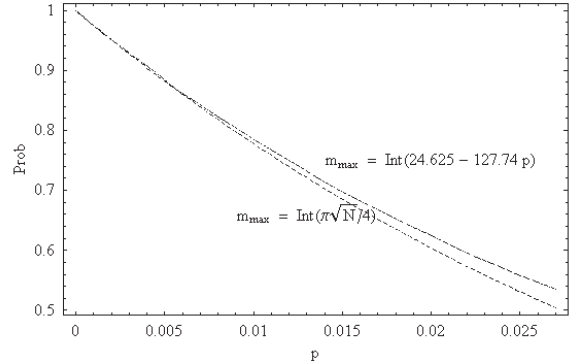


FIG. 7. Probability of the search item as a function of p for different iteration formulae.

In the context of a generalized measurement, we find that probability amplitude of the search item is given by the expression in Eq.(26) and this probability generally reduces in the first maximal iteration. The plots of the probability of the search item as a function of the number of iterations m for various values of p at $N = 128$ and $r = 131$ is plotted in Fig. 8. In the plots, we have chosen a reasonably small value for the parameter, $\epsilon = 0.1$, since under experimental conditions, one do not expect a large value for this parameter. In fact, the results show that for sufficiently small ϵ , it is possible to confine our study to orthogonal Von Neumann measurements.

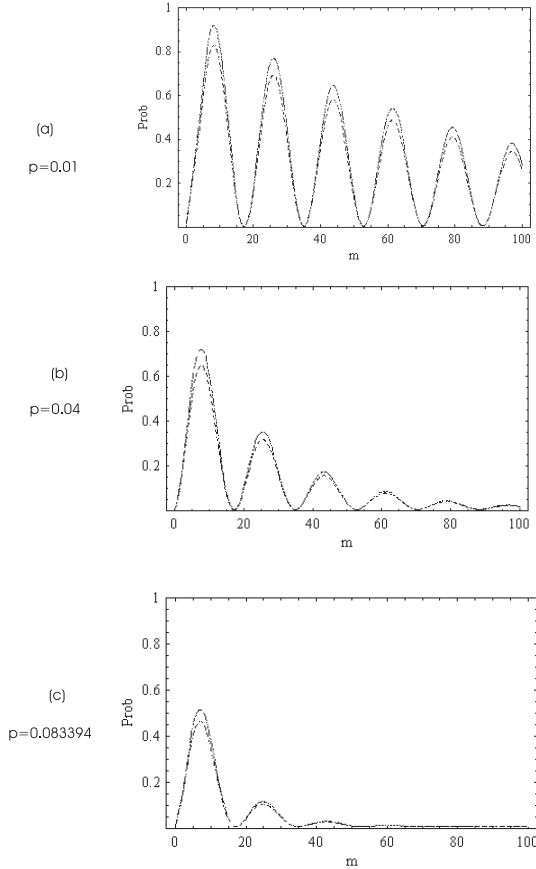


FIG. 8. Probability of the search item as a function of m for $N = 128$, $\epsilon = 0.1$ and (a) $p = 0.01$, (b) $p = 0.04$ and (c) $p = 0.083394$ for orthogonal (bold) and non-orthogonal (dotted) measurements.

VI. DISCUSSION AND CONCLUDING REMARKS

In this paper, we study the effects of the search algorithm analytically and numerically using a simple decoherence model. Several observations are in order. Firstly, we see that the search algorithm can be rendered useless for small degree of decoherence under the depolarizing channel as the database gets larger. Indeed for a database size of about 2 million entries, the degree of decoherence, p , is a factor of 1000 smaller than the corresponding value for a database size of 4. It appears that for large N , the factor varies approximately as \sqrt{N} .

Secondly, as we would expected, decoherence generally diminishes the probability of the search items. With high decoherence, it may not be useful to iterate beyond the first maxima in the probability distribu-

tion of the search item.

Thirdly, the number of iterations needed to maximize a search probability decreases with increasing decoherence. This is a somewhat unexpected result since decoherence ostensibly tends to reduce the probability distribution. Nevertheless, although the associated maximum probability has decreased, it is interesting to note that under a noisy channel, one could in general improve the efficiency by reducing the iterations prior to measurement. It is interesting to note here that the number of iterations (considered as a continuous variable, i.e. prior to taking the integer part) falls off linearly with the degree of decoherence, p . However, if we continue to use the maximum number of iterations in noise-free situation, the difference in probability appears to be less than 10%.

Finally, we note that although the decoherence model that we have employed here is very simple, it certainly provides a wealth of information regarding the behavior of the search algorithm under a noisy environment.

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